Smooth M-estimation in the Current Status Model

Statistik unter einem Dach

Bielefeld

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Illustrating Example: Rubella

- Some facts about Rubella (also called three-days measels or German measels)
  - Once infected, immunity is life-long.
  - Disease can pass unnoticed.
- Interested in stochastic behaviour of variable

$$X \quad = \quad \text{time someone is infected by rubella virus.}$$

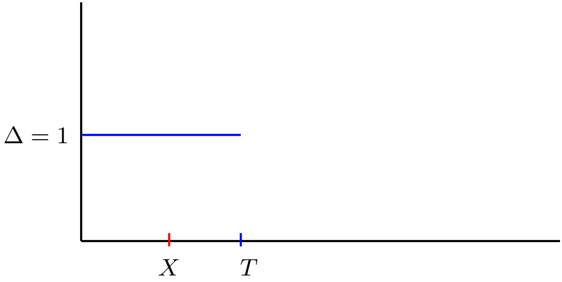
- When people are tested for antibody presence in blood, the vector $Z = (T, \Delta)$ is observed, where

$$T \quad = \quad \text{current age}$$
$$\Delta \quad = \quad \text{current immunization status} \quad = \quad 1_{\{X \le T\}}.$$

General Setup Current Status Model

- **Settings:**

X	variable of interest with **unknown** cdf F_0.
T	censoring variable with (possibly unknown) density g.
$\Delta = 1_{\{X \leq T\}}$	variable indicating whether X hapened before time T or not.
$Z = (T, \Delta)$	observed vector



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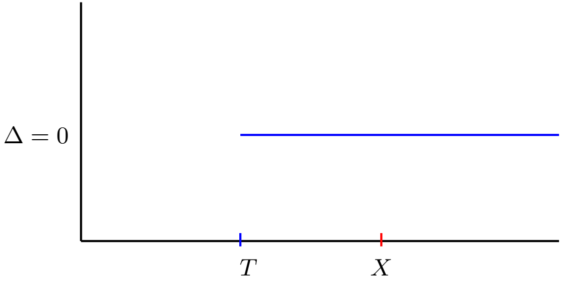

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- **Known:**

Density of observed vector Z is given by

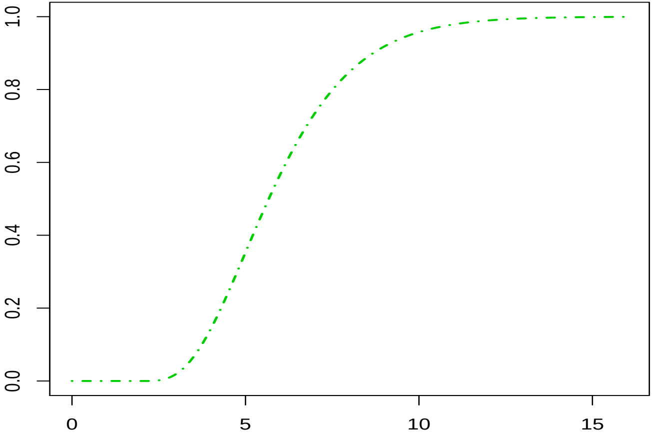
$$f_Z(t, \delta) \quad = \quad g(t)\big\{F(t)\big\}^{\delta}\big\{\big(1 - F(t)\big)\big\}^{(1-\delta)}, \; t > 0$$

- **Goal:**

Estimate the distribution function F_0 and its density f_0, using n iid copies of Z collected during epidemiological studies.

An example

$F_0(x) = 1 - \exp(-x),\ g(t) = \exp(-t),\ n = 250.$



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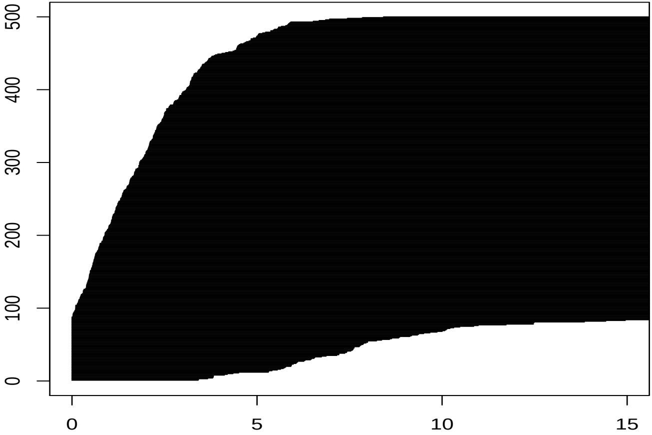

Overview

- Definition of the Estimators
  - Nonparametric Maximum Likelihood Estimator (NPMLE)
  - Maximum Smoothed Likelihood Estimator (MSLE)
  - Smoothed Maximum Likelihood Estimator (SMLE)
- Characterization and Illustration of the Estimators
- Asymptotic Results
  - Distribution Function
  - Density
- Discussion

Some Notation

- K is a kernel density with compact support, i.e. [-1,1], and distribution function $\bar{K}(u) = \int_{-1}^{u} K(x)\,dx$
- h_n is a smoothing parameter (bandwidth) such that $K_{h_n}(u) = 1/h_n K(u/h_n)$, $\bar{K}_{h_n} = \bar{K}(u/h_n)$.
- $\mathbb{G}_n$ is the empirical distribution function (edf) of the T_i's, and $\mathbb{P}_n$ is the edf of the Z_i's.
- g is unknown and estimated by

$$\hat{g}_n(t) = \int K_{h_n}(x-t)\,d\mathbb{G}_n(t),$$

such that $\hat{g}_n(t) = \hat{g}_{n,0}(t) + \hat{g}_{n,1}(t)$

$$\hat{g}_{n,0}(t) = \int (1-\delta) K_{h_n}(x-t)\,d\mathbb{P}_n(t,\delta),$$

$$\hat{g}_{n,1}(t) = \int \delta K_{h_n}(x-t)\,d\mathbb{P}_n(t,\delta).$$

- $\hat{P}_n(t,\delta) = (1-\delta)\hat{G}_{n,0}(t) + \delta\hat{G}_{n,1}(t)$ is a smoothed version of $\mathbb{P}_n$, where

$$\hat{G}_{n,i}(t) = \int_0^t \hat{g}_{n,i}(u)\,du.$$

Estimators for F_0

- Log-likelihood for n iid copies of Z:

$$l(F) = \int \left\{\delta \log F(t) + \left(1-\delta\right) \log \left(1 - F(t)\right)\right\} d\mathbb{P}_n(t,\delta)$$

Nonparametric Maximum Likelihood Estimator (NPMLE):

$$\hat{F}_n = \arg\max_{F \in \mathcal{F}} l(F)$$

Studied by P. Groeneboom & J.A. Wellner (1992), *Information Bounds an Nonparametric Maximum Likelihood Estimation*, New York: Cambridge University Press.

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Nonparametric Maximum Likelihood Estimator (NPMLE):

$$\hat{F}_n = \arg\max_{F \in \mathcal{F}} l(F)$$

- Smoothed log-likelihood for n iid copies of Z:

$$l^S(F) = \int \left\{\delta \log F(t) + \big(1-\delta\big) \log\big(1 - F(t)\big)\right\} d\hat{P}_n(t,\delta)$$

Maximum Smoothed Likelihood Estimator (MSLE):

$$\hat{F}_n^{MS} = \arg\max_{F \in \mathcal{F}} \; l^S(F)$$

Introduced by P.P.B. Eggermont & V.N. LaRiccia (2001), *Maximum Penalized Likelihood Estimation*, New York: Springer-Verlag.

Estimators for F_0

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- Smoothed Maximum Likelihood Estimator (SMLE):

$$\hat{F}_n^{SM}(x) = \int \bar{K}_{h_n}(x-y)\, d\hat{F}_n(y)$$

Estimators for f_0

Based on the relation

$$f_0(x) = \frac{d}{dx} F_0(x)$$

we define the MSLE for $f_0(x)$ as

$$\hat{f}_n^{MS}(x) = \frac{d}{dx} \hat{F}_n^{MS}(x).$$

The SMLE for $f_0(x)$ is defined as

$$\hat{f}_n^{SM}(x) = \frac{d}{dx} \hat{F}_n^{SM}(x) = \int K_{h_n}(x-y)\, d\hat{F}_n(y)$$

Note the relation between $\hat{f}_n^{MS}(x)$ and the estimator $\hat{g}_n(x)$, defined by

$$\hat{g}_n(x) = \int K_{h_n}\big(x-y\big)\, d\mathbb{G}_n(y).$$

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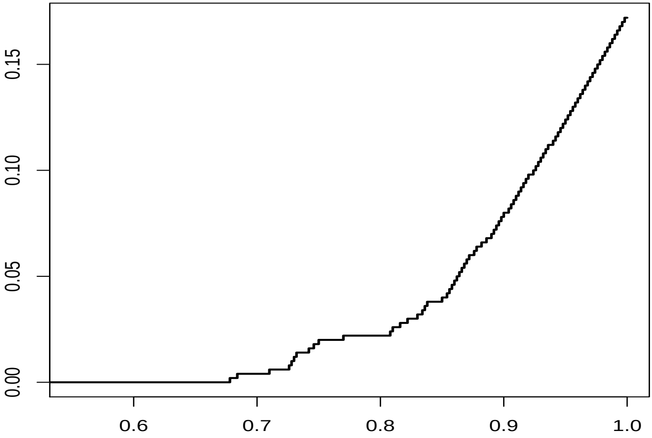
Proposition (Groeneboom & Wellner (1992))

Let H^* be the convex minorant of the cusum diagram (CSD) consisting of the points

$$t \mapsto \left(\mathbb{G}_n(t), \int_{u\in[0,t]} \delta \, d\mathbb{P}_n(u,\delta)\right), \; t \in [0, T_{(n)}]$$

Let $\hat{F}_n(t)$ be the right-continuous slope of H^*. Then $\hat{F}_n$ maximizes $l(F)$ over the class of all distribution functions.

Illustration of $\hat{F}_n$



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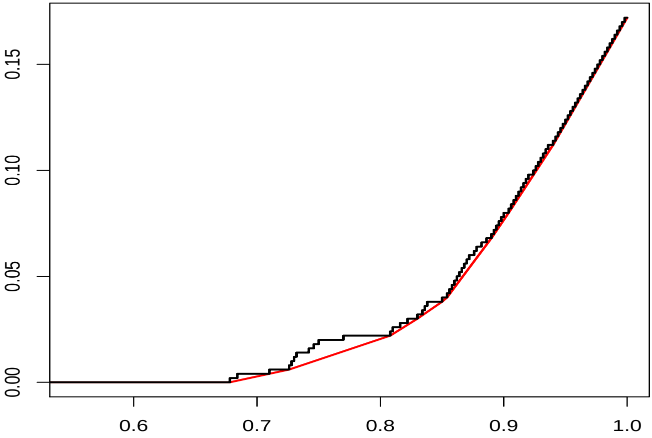

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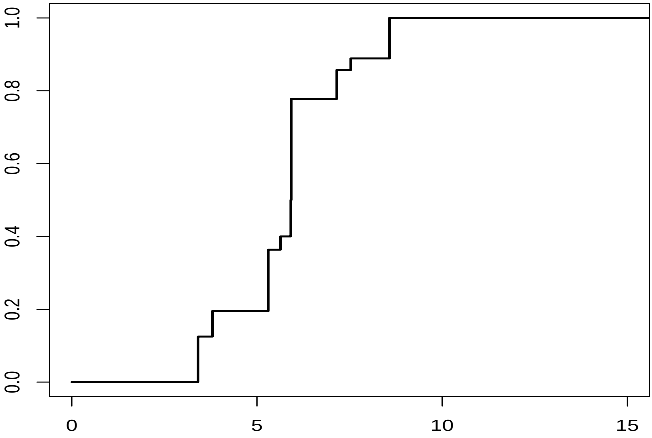

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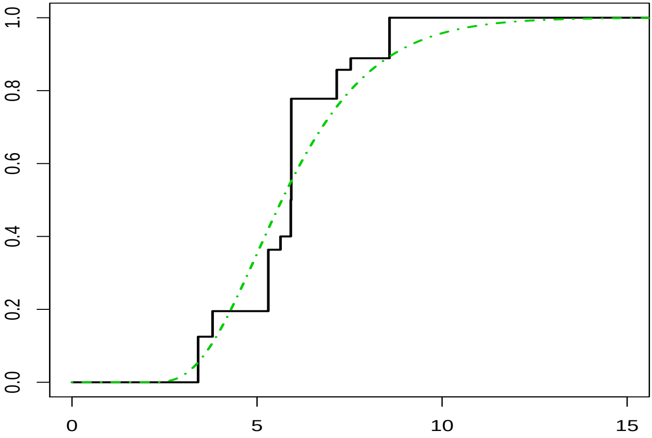

Naive MSLE

$$l^S(F) = \int \{\delta \log F(t) + (1-\delta)\log(1-F(t))\} d\hat{P}_n(t,\delta)$$

Pointwise maximization of the integrand, by differentiating w.r.t. $F(t)$, leads to

$$\hat{F}_n^{\text{naive}}(t) = \frac{\hat{g}_{n,1}(t)}{\hat{g}_n(t)}, \quad \hat{f}_n^{\text{naive}}(t) = \frac{d}{dt}\frac{\hat{g}_{n,1}(t)}{\hat{g}_n(t)}.$$

Done by M.H. Maathuis (2003), Nonparametric Maximum Likelihood Estimator for Bivariate Censored Data in Continuous Time, Master's Thesis, Delft University of Technology.

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Advantage:

- Easy to compute numerically.
- Easy to derive asymptotics.

Disadvantage:

- Might decrease locally.

Theorem

Consider the following parameterized curve in $\mathbb{R}_+^2$, which we call the continuous cusum diagram (CCSD):

$$t \mapsto \left(\hat{G}_n(t), \hat{G}_{n,1}(t)\right), \; t \in [0, \tau].$$

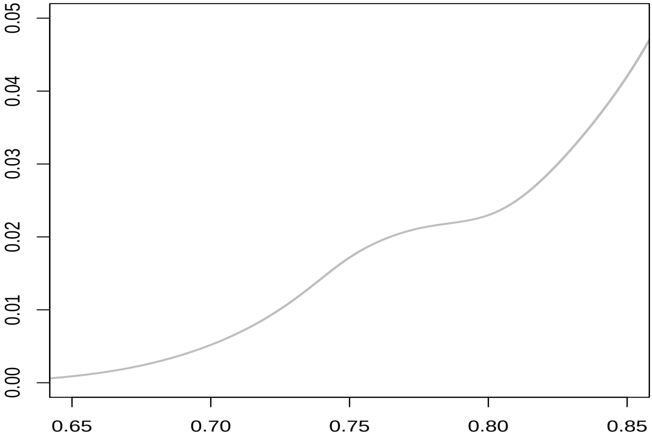
Let $\hat{F}_n^{MS}(t)$ be the right-continuous slope of the lower convex hull of the CCSD, evaluated at the point with x-coordinate $\hat{G}_n(t)$. Then

(i) *$\hat{F}_n^{MS}$ maximizes $l^S(F)$ over the class of all sub-distribution functions.*

(ii) *$\hat{F}_n^{MS}$ is the unique maximizer of $l^S(F)$.*

Illustration of $\hat{F}_n^{MS}$

$h = 0.4,\ K(u) = \frac{35}{32}(1-u^2)^3 1_{[-1,1]}(u).$



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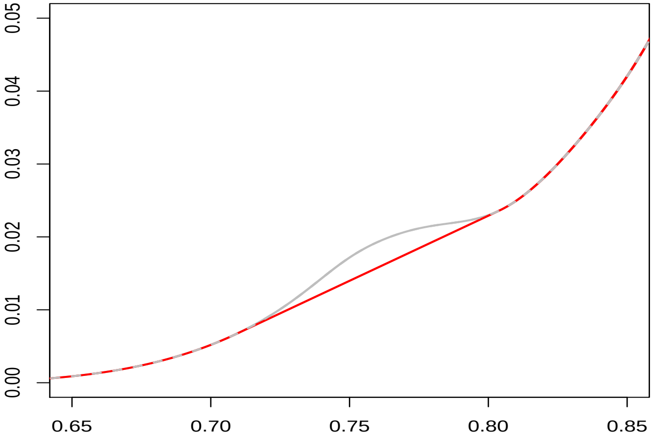

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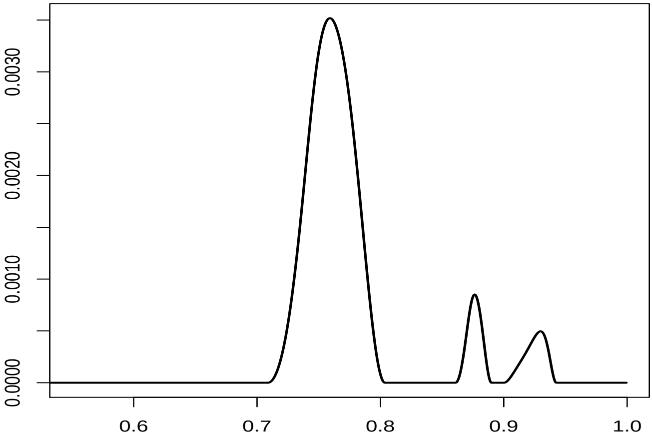

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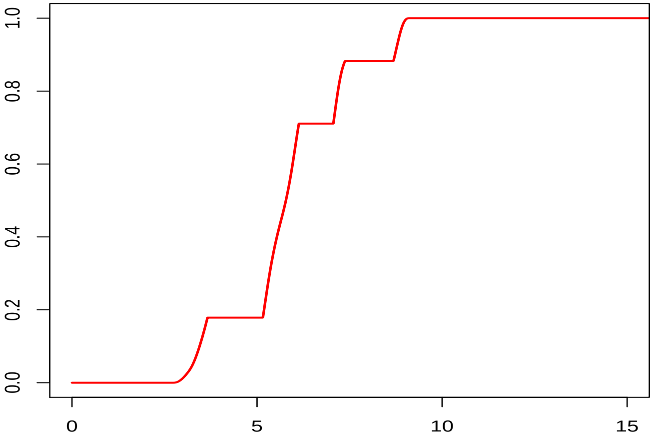

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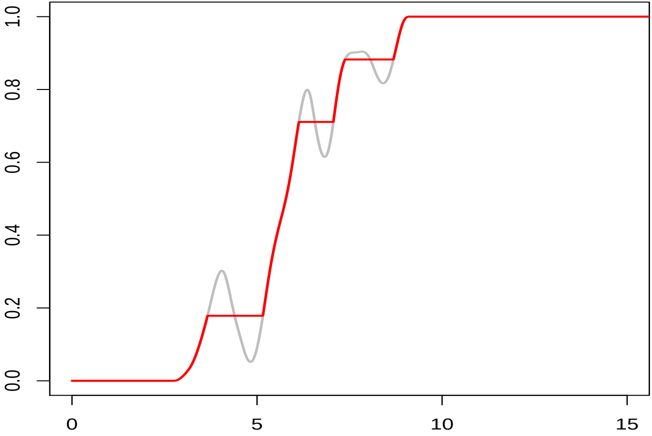

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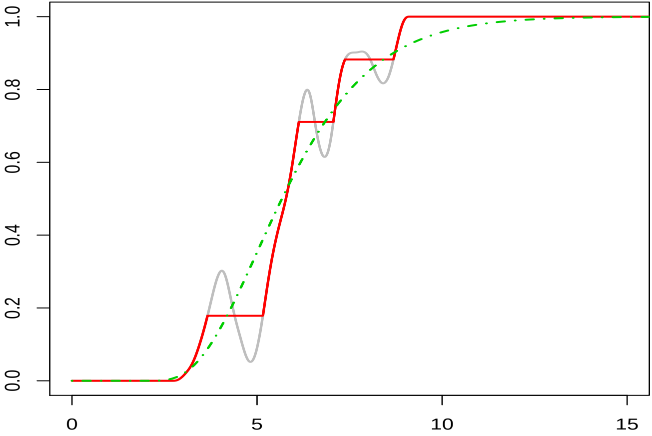

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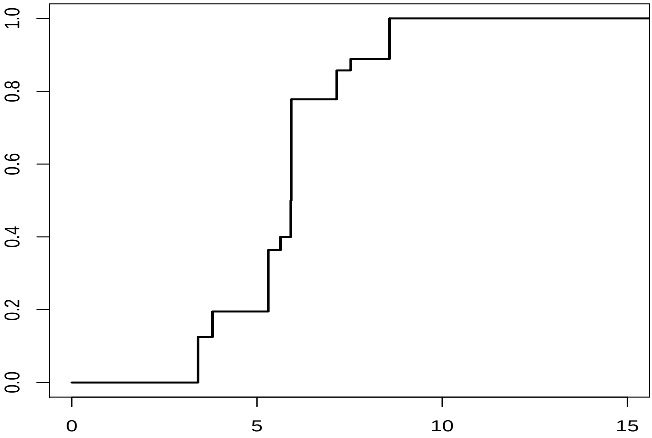

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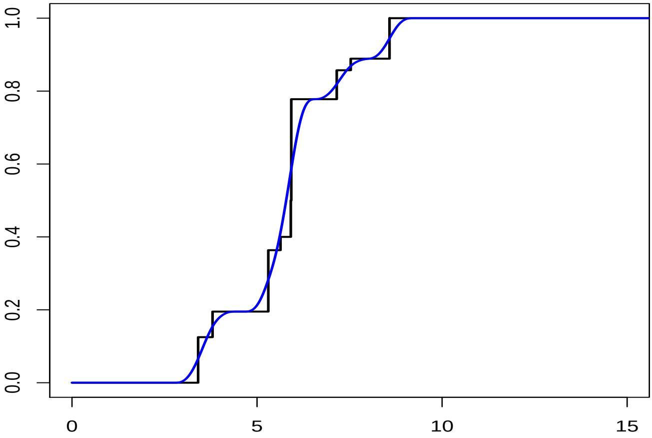

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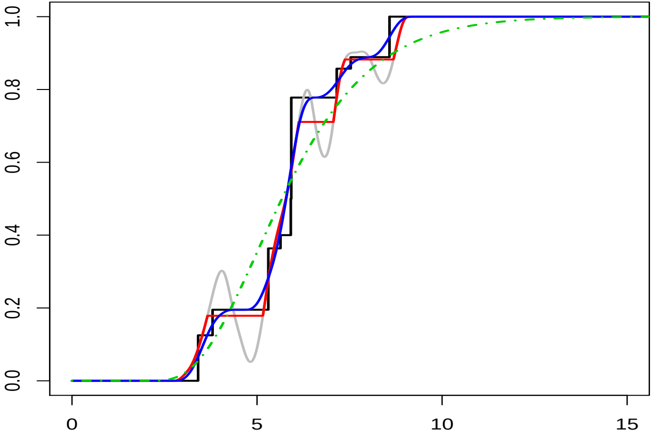
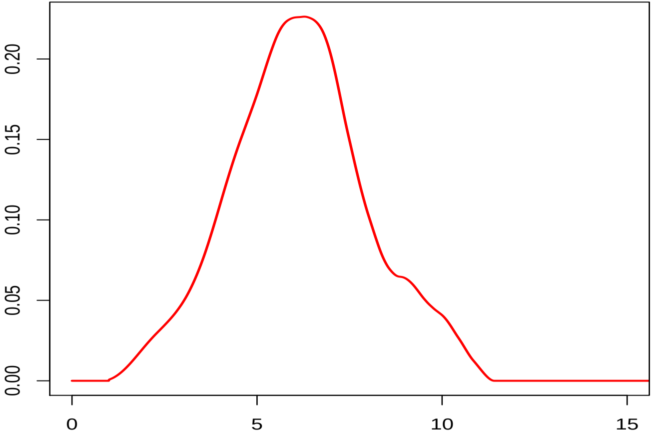


Illustration of $\hat{f}_n^{MS}$

$h = 2.0,\ K(u) = \frac{35}{32}(1-u^2)^3 1_{[-1,1]}(u)$



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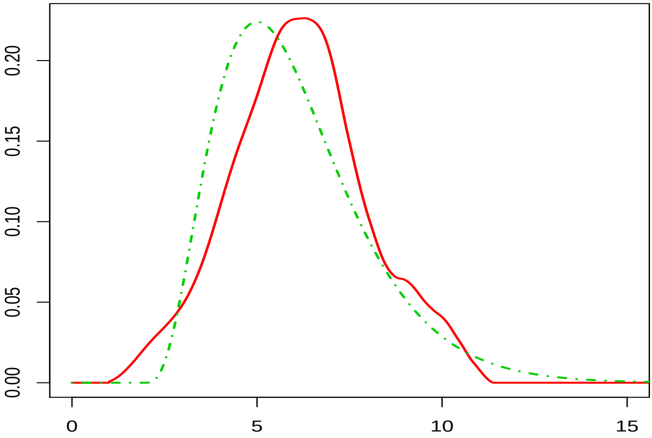

Illustration of $\hat{f}_n^{SM}$

Introduction

The Estimators

Characterization

Asymptotics
distr. function
density

Discussion

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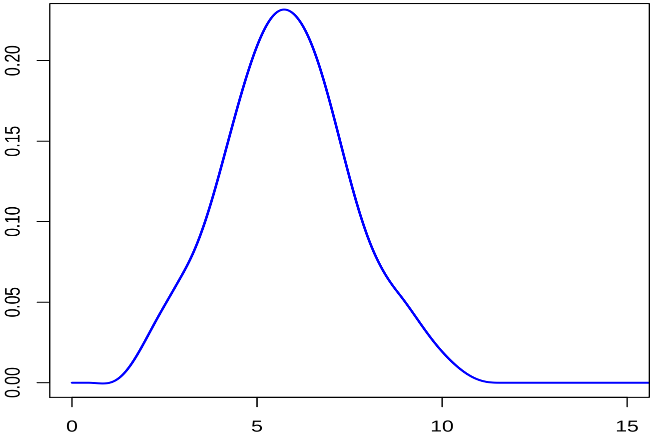

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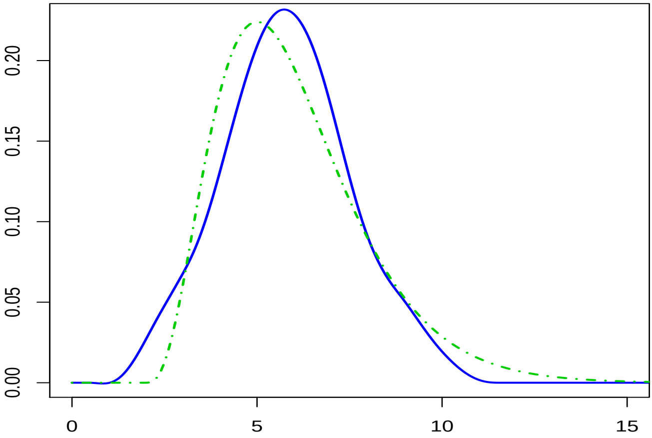

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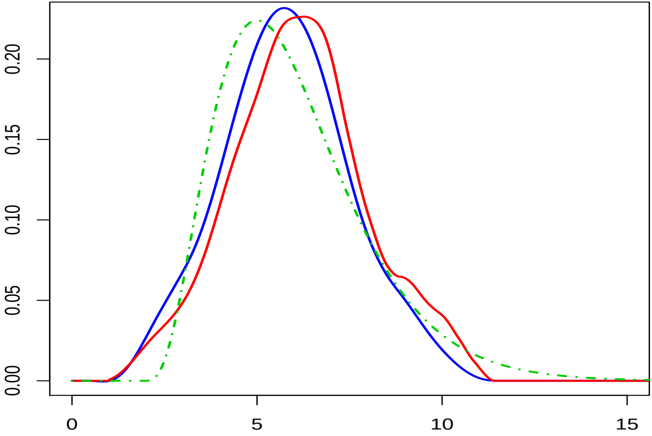

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Theorem (Groeneboom & Wellner(1992))

Let x be such that $0 < F_0(x), G(x) < 1$. Then we have, under certain conditions,

$$\left(\frac{F_0(x)\big(1 - F_0(x)\big) f_0(x)}{2g(x)}\right)^{-1/3} n^{1/3}\big(\hat{F}_n(x) - F_0(x)\big) \rightsquigarrow 2Z,$$

where Z is the last time where the standard two-sided Brownian motion minus the parabola $y(t) = t^2$ reaches its maximum.

Asymptotics for $\hat{F}_n^{\text{naive}}$

Theorem

Let $h_n = cn^{-\alpha}$ be the smoothing parameter. Under certain conditions it holds that $\forall\ \varepsilon > 0,\ \forall\ M > \varepsilon$ and $\forall\ \alpha \in (0, 1/3)$:

$$P\big(\hat{F}_n^{\text{naive}} \text{ is monotonically increasing on } [\varepsilon, M]\big) \longrightarrow 1.$$

Proof.

Follows from uniform consistency of estimates $\hat{g}_n(t)$, $\hat{g}_{n,1}(t)$, $\hat{g}_n'(t)$ and $\hat{g}_{n,1}'(t)$. □

Corollary

Let $h_n = cn^{-\alpha}$ be the smoothing parameter. Then for all $x > 0$ and $\forall\ \alpha \in (0, 1/3)$ it holds that the estimators $\hat{F}_n^{MS}(x)$ and $\hat{F}_n^{\text{naive}}(x)$ are asymptotically equivalent.

Asymptotics for $\hat{F}_n^{MS}$

Theorem

For fixed $x > 0$ and $h_n = cn^{-1/5}$ we have

$$n^{2/5}\big(\hat{F}_n^{MS}(x) - F_0(x)\big) \quad \rightsquigarrow \quad \mathcal{N}(\mu_F, \sigma_F^2)$$

where

$$\mu_F = \frac{1}{2}c^2 m_2(K) f_0'(x) + c^2 m_2(K)\frac{f_0(x)g'(x)}{g(x)},$$

$$\sigma_F^2 = \frac{F_0(x)\big(1 - F_0(x)\big)}{cg(x)} \int K(y)^2\, dy.$$

This also holds if we replace $\hat{F}_n^{MS}$ by $\hat{F}_n^{\mathrm{naive}}$.

Proof.

A combination of Lindeberg-Feller Central Limit Theorem and the Delta-method. □

Theorem

For fixed $x > 0$ and $h_n = cn^{-1/5}$ we have

$$n^{2/5}\big(\hat{F}_n^{SM}(x) - F_0(x)\big) \quad \rightsquigarrow \quad \mathcal{N}(\nu_F, \tau_F^2),$$

where

$$\nu_F = \frac{1}{2}c^2 m_2(K) f_0'(x)$$

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Proof.

Based on Empirical Process Theory. □

Theorem

For fixed $x > 0$ and $h_n = cn^{-1/7}$ we have

$$n^{2/7}\big(\hat{f}_n^{MS}(x) - f_0(x)\big) \quad \rightsquigarrow \quad \mathcal{N}(\mu_f, \sigma_f^2)$$

where

$$\begin{aligned}\mu_f \quad &= \quad \frac{1}{2}c^2 m_2(K) f_0''(x) \\ &\qquad + \frac{m_2(K)c^2}{g(x)}\left(g''(x)f_0(x) + g'(x)f_0'(x) - \frac{g'(x)^2 f_0(x)}{g(x)}\right) \\ \sigma_f^2 \quad &= \quad \frac{F_0(x)\big(1 - F_0(x)\big)}{c^3 g(x)} \int K'(y)^2\, dy.\end{aligned}$$

This also holds if we replace $\hat{f}_n^{MS}$ by $\hat{f}_n^{\text{naive}}$.

Proof.

A combination of Lindeberg-Feller Central Limit Theorem and the Delta-method. □

Asymptotics for $\hat{f}_n^{SM}$

Theorem

For fixed $x > 0$ *and* $h_n = cn^{-1/7}$ *we have*

$$n^{2/7}\big(\hat{f}_n^{SM}(x) - f_0(x)\big) \quad \rightsquigarrow \quad \mathcal{N}(\nu_f, \tau_f^2),$$

where

$$\nu_f \quad = \quad \frac{1}{2}c^2 m_2(K) f_0''(x),$$

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Remarks:

- Taking $h_n = 0$ in both methods for estimating F_0 yields the NPMLE $\hat{F}_n$.
- The asymptotic bias of the estimators $\hat{F}_n^{MS}$ and $\hat{F}_n^{SM}$ differ, but it does not hold that one is always better than the other. The same holds for the estimators $\hat{f}_n^{MS}$ and $\hat{f}_n^{SM}$.
  Also seen in P.N. Patil et al (1994), Some heuristics of kernel based estimators of ratio functions, *Nonparametric Statistics*, **4**, 203–209.
- The asymptotic variance of the estimators $\hat{F}_n^{MS}$ and $\hat{F}_n^{SM}$ are equal, as are the asymptotic variances of $\hat{f}_n^{MS}$ and $\hat{f}_n^{SM}$.

Future work:

- Bandwidthselection methods.
- Current Status Model with Continuous Mark Variable
- Other bivariate censoring Models under the assumption of Stochastic Ordering

Thank you for your attention!

References

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